Invariance in Structural Models

Gadi S. Perets*
Mathematics Department, Universite de Lyon 1 and UMR 5208, CNRS
Eran Yashiv†
Eitan Berglas School of Economics, Tel Aviv University,
CfM (London School of Economics), and CEPR

November 30, 2021

Abstract

Invariance plays a key role in structural models, often used for the design and evaluation of policy plans. This paper shows how the algebraic method of Lie symmetries of differential equations can derive the full set of conditions for this invariance. In doing so, key problems in the formulation of such models are resolved.

We demonsrate an application of this algebra to a widely-used Macro-Finance model. We then outline a diverse set of models at the research frontier, which would be amenable to similar analysis, thereby providing a road map for a potentially important new literature.

Key words: invariance, structural models, Lie symmetries, differential equations, policy design and evaluation.

JEL codes: C10, C50, C60, D01, D11, E21, G11

*E-mail: gadi@math.univ-lyon1.fr

†*Corresponding author*. E-mail: yashiv@tauex.tau.ac.il; e.yashiv@lse.ac.uk

Invariance in Structural Models[1]

1 Introduction

This paper shows how an algebraic tool, Lie symmetries of differential equations, may be used in formulating structural models in Economics. Such models are heavily used for the design and evaluation of policy plans. For the latter one needs structural models, featuring stable relations, optimal behavior by agents, and parameters which are policy invariant. There are, however, well known hurdles and problems when coming to formulate such models: they may not have a closed form solution; theory typically provides limited guidance for the specification of functional forms of the model; it may be difficult to select a model out of a number of formulations, which derive from a single theoretical framework and fit the data well, but imply different counterfactual predictions; and the full set of invariant policy parameters may not be known. The algebraic technique of Lie symmetries of differential equations provides a solution to these problems. It allows for the derivation of the complete set of invariance conditions in economic optimization problems. This greatly facilitates the formulation of structural models, which can then be taken to the data.

A symmetry is an invariance under transformation. This concept is usually known for the case of the invariance of functions, the homothetic utility or production functions being the most well-known special cases. In this paper we employ Lie symmetries, which are symmetries of differential equations, and do so in the context of economic optimization problems. Importantly, the symmetries provide the solutions of these equations, or generate rich information with respect to the properties of the solutions, even when no closed-form solutions exist. They derive the full set of conditions whereby the solutions remain invariant.

We demonstrate how this algebraic method works through an application, using a key model in Finance, which is also isomorphic to benchmark models used in Macroeconomics. The symmetries impose restrictions on the model and define functional forms, which can then be estimated. Thus, the symmetries define the objects for structural econometric estimation to be used for policy design and evaluation. The emerging functional forms also facilitate aggregation and the construction of equilibrium models.

There is an abundance of optimality equations in Economics, which are PDE with

[1] *We thank Tzachi Gilboa, Jim Heckman, and Ryuzo Sato for useful conversations. Any errors are our own.*

complicated structures,[2] that can be constructively explored using the Lie symmetries methods presented in this paper. We discuss key models at the research frontier, which would be amenable to such analysis, focusing on what we see as the most promising and important ones currently. This provides a road map for a potentially important new literature.

The paper proceeds as follows: in Section 2 we briefly discuss seminal and survey papers in the structural econometrics literature and cite the key relevant references in the Lie symmetries literature. In Section 3 we relate the idea of invariance of optimal behavior in an economic model, which can be subject to policy plans, with invariant structural models in empirical studies, which may be used to design and evaluate policy. In Section 4 we elucidate the mathematical concept of Lie symmetries of differential equations. In Section 5 we discuss the implementation of this algebraic methodology, including a sub-section (5.4) which presents the implications for the study of economic policy and discusses the connections of the model and its symmetries to structural econometrics. Section 6 discusses specific models at the research frontier amenable to this analysis. Section 7 concludes.

2 The Literature

This paper relates to two strands of literature.

Structural econometrics for policy design and evaluation. Econometric work with invariant structures, in the context of causal analysis and policy design and evaluation, is reviewed and discussed in Heckman and Vytlacil (2007, see Section 4), Heckman (2008), Heckman and Pinto (2014), Athey and Imbens (2017), and Low and Meghir (2017). The seminal work on these topics can be traced back to Frisch (1938) and Haavelmo (1943),[3] with important further developments in the work of the Cowles Commission (and, later, Foundation) in the 1940s, 1950s and 1960s. In Macroeconomics and Finance, a major turning point was associated with the Lucas (1976) critique of reduced-form em-

[2]For a review of PDE equations in Macroeconomics see Achdou, Buera, Lasry, Lions, and Moll (2014).

[3]An interesting fact to note is a Norwegian or an Oslo University "connection": the fundamental mathematical work on symmetries of differential equations was undertaken by Sophus Lie (1842-1899), a Norwegian, who got his PhD at the University of Christiania, now Oslo, in 1871. The econometric approach making use of invariant structures was proposed by Ragnar Frisch (1895-1973), a Norwegian, who got his PhD at the University of Oslo, in 1926. Frisch was editor of Econometrica for over 20 years (1933-1954) and won the 1969 Nobel prize (shared with Jan Tinbergen). Important developments of the econometric work on this topic were introduced by Trygve Haavelmo (1911-1999), a Norwegian, who was professor of Economics and Statistics at the University of Oslo for more than 30 years (1948–79). He was awarded the Nobel prize in 1989.

pirical models, which was built on these early insights. This approach was advanced by the development of structural estimation and Rational Expectations Econometrics, mostly associated with the work of Sargent and Hansen (see, for example, Hansen and Sargent (1980) and Hansen (2014)). The relationship of this econometric literature with the current paper is that we show how the application of Lie symmetries to economic optimization problems yields restrictions on the model, which can then be estimated using structural econometrics. In this context the following definition by Heckman and Vytlacil (2007, p.4848) is pertinent:

> "A more basic definition of a system of structural equations, and the one featured in this chapter, is a system of equations invariant to a class of modifications. Without such invariance one cannot trust the models to forecast policies or make causal inferences."

Lie Symmetries. In the Mathematics literature, Olver (1993, 1999, 2012) offers extensive formal discussions of the concept and use of Lie symmetries, including applications. In particular, the prolongation methodology, which is a key one and is presented in sub-section 5.2.1 below, is discussed at length. There are a number of software codes for symbolic analysis of Lie symmetries; see reviews in Filho and Figueiredo (2011) and Vu, Jefferson and Carminati (2012) and applications in Kaibe and O'Hara (2019).

Pioneering contributions to economic applications of *Lie algebra* were made by Sato (1981) and Sato and Ramachandran (1990), with further developments in Sato and Ramachandran (2014) and references therein. For applications in Finance, see Sinkala, Leach, and O'Hara (2008 a,b) and Kaibe and O'Hara (2019). We are not aware, though, of applications of *Lie symmetries* for the modelling of structural models with invariance in Economics as this paper does.

3 Invariance of Optimal Economic Behavior and Structural Models

We connect the idea of invariance of optimal behavior in an economic model, which can be subject to policy plans, with invariant structural models to be subsequently used in empirical studies, which may serve for policy design and evaluation.

We begin with the latter. Discussing structural analysis in Econometrics, Heckman and Vytlacil (2007) propose the following definitions and examples.

p. 4827. The traditional model of econometrics is the "all causes" model. It writes outcomes as a deterministic mapping of inputs to outputs:

$$y(s) = g_s(x, u_s) \tag{1}$$

where x and u_s are fixed variables specified by the relevant economic theory. This notation allows for different unobservables u_s to affect different outcomes. Đ is the domain of the mapping g_s : Đ$\rightarrow$ R^y, where R^y is the range of y. There may be multiple outcome variables. All outcomes are explained in a functional sense by the arguments of g_s in (1). If we model the ex post realizations of outcomes, it is entirely reasonable to invoke an all causes model. Ex post, all uncertainty has been resolved. Implicit in the definition of a function is the requirement that g_s be "stable" or "invariant" to changes in x and u_s. The g_s function remains stable as its arguments are varied. Invariance is a key property of a causal model.

pp. 4846-7. Parameters of a model or parameters derived from a model are said to be policy invariant with respect to a class of policies if they are not changed (are invariant) when policies within the class are implemented...

More generally, policy invariance for f, g or $\{f_s, g_s\}_{s \in S}$ requires for a class of policies $P_A \subseteq P$,

(PI-5). The functions f, g or $\{f_s, g_s\}_{s \in S}$ are the same for all values of the arguments in their domain of definition no matter how their arguments are determined, for all policies in P_A...

In the econometric approach to policy evaluation, the analyst attempts to model how a policy shift affects outcomes without reestimating any model. Thus, for the tax and labor supply example..., with labor supply function $h_s = h(w(1-s), x, u_{s)},$ it is assumed that we can shift tax rate s without affecting the functional relationship mapping $(w(1-s), x, u_s)$ into h_s . If, in addition, the support of $w(1-s)$ under one policy is the same as the support determined by the available economic history, for a class of policy modifications (tax changes), the labor supply function can be used to accurately predict the outcomes for that class of tax policies.

In most cases of interest, the functions of the kind discussed above by Heckman and Vytlacil (2007), are derived from an optimization problem. Thus, for example, the labor hours supply function emerges from maximization of worker utility, subject to constraints. When these optimization problems feature differential equations – such as the HJB equation discussed in the main example of the current paper – Lie symmetries can be used to derive the *full set* of solutions to the optimization problem. The functions in these solutions satisfy the invariance properties discussed above. Hence structural models featuring the invariance needed for structural econometrics can be derived using Lie symmetries.

Olver (1993, p.93) offers the following formal definition.[4]

> DEFINITION 2.23. Let $\mathcal{L}$ be a system of differential equations. A *symmetry group* of the system $\mathcal{L}$ is a local group of transformations G acting on an open subset M of the space of independent and dependent variables for the system with the property that whenever $p = f(x)$ is a solution of $\mathcal{L}$, and whenever $g \cdot f$ is defined for $g \in G$, then $p = g \cdot f(x)$ is also a solution of the system.[5]

In economic applications, policy, such as the tax example givan above, may underlie the transformations of G, and the idea of 'policy modification' in an invariant structural model maps to a symmetry group. To be more concrete, consider h_s (in the Heckman and Vytlacil (2007) text above) as the function in question. It is defined over $w(1-s)$. Hence g could be translation such that

$$\begin{array}{ccc} (w(1-s)) & \longmapsto & (w(1-\alpha s)), \\ \alpha & \in & \mathbb{R} \end{array}$$

The shift in s can be modelled as $h_{s_1} = g \cdot h_{s_0}$.Thus, by the Lie symmetries definition,

---

[4]For exact definitions of the key mathematical concepts used here, see Appendix A.

[5]Note that:

a. $g \cdot f$ means the transformation g on the function f, thereby expressing the action of the symmetry group G on the function f. The dot here is **not** the multiplication sign.

b. A transformation group acting on a smooth manifold M is determined by a Lie group G and smooth map $\Phi : G \times M \to M$, denoted by $\Phi(g, x) = g \cdot x$, which satisfies the following:

$e \cdot x = x$,**cHECK**

$g \cdot (h \cdot x) = (g \cdot h) \cdot x$,

for all $x \in M, g \in G$.

A Lie group G is a smooth manifold which is also a group, such, that the group multiplication $(g, h) \longmapsto g \cdot h$ and inversion $g \longmapsto g^{-1}$ define smooth maps.

c. The action is taking place on the space of optimal solutions, not of the variables. This mean that if $f(s)$ is a function in this space, and g an element of the symmetry group, G, then $g \cdot f$ will be an optimal solution function in this space.

if

$$h_{s_0} = h(w(1-s_0), x, u_s)$$

is a solution (to the optimality conditions of the labor supply problem) then

$$h_{s_1} = g \cdot h(w(1-s_1), x, u_s)$$

- is also a solution, where $s_1 = \alpha s_0$.

Similarly, g can be an additive shift or one of many, much more complex, shifts. The issue is to find what is the symmetry group G that satisfies these relations and this is where Lie symmetries analysis comes into play. Note that if we find such a symmetry we know under what conditions "we can shift tax rate s without affecting the functional relationship mapping $(w(1-s), x, u_s)$ into h_s." Because of an iff property we discuss below, we then know all of the relevant conditions.

We turn now to describe Lie symmetries and show how they work.

4 Lie Symmetries

We briefly introduce the mathematical concept of Lie symmetries of differential equations.

Lie symmetries of differential equations are the transformations which leave the space of solutions invariant. We begin by explaining the concept of invariance of differential equations, culminating by the derivation of the prolongation equation, which is key in deriving the Lie symmetries of a differential equations system. In making the exposition here we are attempting to balance two considerations: the need to explain the mathematical derivation and the constraint that an overload of mathematical concepts may be burdensome to the reader.

Consider the differential equation:

$$L(t, x, y, p) = 0 \tag{2}$$

where $x = x(t), y = y(x), p = \frac{dy}{dx}$ and t is time.

The transformation:

$$
\begin{aligned}
x' &= \phi(x, y, t) \\
y' &= \psi(x, y, t)
\end{aligned} \tag{3}
$$

implies the transformation of the derivative $p = \frac{dy}{dx}$ to:

$$
p' = \frac{dy'}{dx'} = \frac{\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy}{\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy} = \frac{\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}p}{\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}p} \tag{4}
$$

The differential equation (2) will be invariant under the transformation $x \to x'$ and $y \to y'$ (i.e., one integral curve is mapped to another) if and only if it is invariant under:

$$
\begin{aligned}
x' &= \phi(x, y, t) \\
y' &= \psi(x, y, t) \\
p' &= \chi(x, y, p, t)
\end{aligned} \tag{5}
$$

The condition for transformation (5) to leave the differential equation (2) invariant is:

$$
H'L \equiv \xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + \eta'\frac{\partial L}{\partial p} = 0 \tag{6}
$$

where:

$$
\begin{aligned}
H &= (\frac{\partial\phi}{\partial t})_0\frac{\partial}{\partial x} + (\frac{\partial\psi}{\partial t})_0\frac{\partial}{\partial y} \\
&= \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \\
\xi &\equiv (\frac{\partial\phi}{\partial t})_0 \;\; \eta \equiv (\frac{\partial\psi}{\partial t})_0 \\
\eta' &\equiv \frac{\partial\eta}{\partial x} + (\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x})p - \frac{\partial\xi}{\partial y}p^2
\end{aligned}
$$

and the subscript 0 denotes the derivative at $t = 0$; the notation $\frac{\partial}{\partial x}$ is used for a directional derivative i.e., the derivative of the function in the direction of the relevant coordinate axis, assuming space is coordinated. For this and other technical concepts, see Chapter 1 in Sato and Ramachandran (1990).

To see the intuition underlying equation (6) consider the invariance of a function (a generalization of homotheticity) rather than that of a differential equation: a function $f(x,y)$ is invariant under a transformation $x \to x'$ and $y \to y'$ if $f(x,y) = f(x',y')$. Using a Taylor series and infinitesimal transformations we can write:

$$f(x',y') = f(x,y) = f(x,y) - sHf + \frac{s^2}{2}H^2 f + ...$$

It is evident that the necessary and sufficient condition for invariance in this case is:

$$Hf = 0 \tag{7}$$

Equation (6) is the analog of equation (7) for the case of a differential equation. It is called the prolongation equation and it is linear in ξ and η.[6] Finding the solution to it gives the infinitesimal symmetries from which the symmetries of the differential equation itself may be deduced.

The power of this theory lies in the notion of infinitesimal invariance: one can replace complicated, possibly highly non-linear conditions for invariance of a system by equivalent linear conditions of infinitesimal invariance. This is analogous to the use of derivatives of a function at a point to approximate the function in the neighborhood of this point. Likewise, the infinitesimal symmetries are "derivatives" of the actual symmetries and the way to go back from the former to the latter is through an exponentiation procedure.

The Lie symmetries are derived by calculating their infinitesimal generators, which are vector fields on the manifold composed of all the invariance transformations. Finding these generators is relatively easy, as it is more of an algebraic calculation, while finding the invariance transformations directly amounts more to an analytic calculation. After finding the infinitesimal generators, we "exponentiate" them to get the actual invariant transformations.

A general infinitesimal generator is of the form:

$$v = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{8}$$

We determine all the possible functions ξ, τ, ϕ through the prolongation equation, which puts together all the possible constraints on the functions ξ, τ, ϕ. We show how

[6] The full prolongation formula is presented in Olver (1993) in Theorem 2.36 on page 110. This term comes from the idea of "prolonging" the basic space, representing the independent and dependent variables under consideration, to a space, which also represents the various partial derivatives occurring in the system.

this is implemented in the next Section, when applying Lie symmetries to a fundamental optimization problem in Economics.

5 An Implementation Example

To show how this methodology works, we implement Lie symmetries in the Merton (1969, 1971) model of consumer-investor choice. We chose this model for our implemetation example for two reasons: first, it is a fundamental model in Macroeconomics[7] and Finance, featuring a differential equation defining optimal behavior at its core. It has straightforward taxation policy aspects, and it is amenable to structural econometrics. Second, the model and its solution are very familiar to economists. Thus the idea is not to propose a new model or to discuss a model without a closed-form solution, but rather to demonstrate the implementaiton of Lie symmetries and discuss the emerging insights.

5.1 A Macro-Finance Model

We briefly present the main ingredients of the consumer-investor optimization problem under uncertainty as initially formulated and solved by Merton (1969, 1971)[8]. A key point, which merits emphasis, is that in what follows we do not just show that this model can be solved in a different way. Rather, we use Lie symmetries to solve it and derive the full set of invariance conditions.

The essential problem is that of an individual who chooses an optimal path of consumption and portfolio allocation. The agent begins with an initial endowment and during his/her lifetime consumes and invests in a portfolio of assets (risky and riskless). The goal is to maximize the expected utility of consumption over the planning horizon and a "bequest" function defined in terms of terminal wealth.

Formally the problem may be formulated in continuous time, using Merton's notation, as follows: denote consumption by C, financial wealth by W, time by t (running from 0 to T), utility by U, and the bequest by B. There are two assets used for investment,[9] one of which is riskless, yielding an instantaneous rate of return r. The other

[7] The Ramsey model and the stochastic growth model, which underlie business cycle modelling, are basically variants of this model.

[8] For a discussion of developments since the initial exposition of these papers see Merton (1990, Chapter 6), Duffie (2003), and Skiadas (2009, Chapters 3,4 and 6).

[9] The problem can be solved with n risky assets and one riskless asset. As in Merton (1971) and for the sake of expositional simplicity, we restrict attention to two assets. Our results apply to the more general case as well.

asset is risky, its price P generated by an Ito process as follows:

$$\frac{dP}{P} = \alpha(P,t)dt + \sigma(P,t)dz \tag{9}$$

where α is the instantaneous conditional expected percentage change in price per unit time and σ^2 is the instantaneous conditional variance per unit time.

The consumer seeks to determine optimal consumption and portfolio shares according to the following:

$$\max_{(C,w)} E_0 \left\{ \int_0^T U[C(t),t]dt + B[W(T),T] \right\} \tag{10}$$

subject to

$$dW = w(\alpha - r)Wdt + (rW - C)dt + wW\sigma dz \tag{11}$$

$$W(0) = W_0 \tag{12}$$

where w is the portfolio share invested in the risky asset. All that needs to be assumed about preferences is that U is a strictly concave function in C and that B is concave in W. See Kannai (2004, 2005) for discussions of utility function concavity as expressing preference relations.

Merton (1969, 1971) applied stochastic dynamic programming to solve the above problem. In what follows we repeat the main equations; see Sections 4-6 of Merton (1971) for a full derivation.

Define:

(i) An "indirect" utility function:

$$J(W,P,t) \equiv \max_{(C,w)} E_t \left\{ \int_t^T U(C,s)ds + B[W(T),T] \right\} \tag{13}$$

where E_t is the conditional expectation operator, conditional on $W(t) = W$ and $P(t) = P$.

(ii) The inverse marginal utility function:

$$G \equiv [\partial U/\partial C]^{-1} \equiv U_C{}^{-1}(C) \tag{14}$$

The following notation will be used for partial derivatives: $U_C \equiv \partial U/\partial C$, $J_W \equiv \partial J/\partial W$, $J_{WW} \equiv \partial^2 J/\partial W^2$, and $J_t \equiv \partial J/\partial t$.

A sufficient condition for a unique interior maximum is that $J_{WW} < 0$ i.e., that J be strictly concave in W.

Merton assumes "geometric Brownian motion" holds for the risky asset price, so α and σ are constants and prices are distributed log-normal. In this case J is independent of P, i.e., $J = J(W, t)$.

Time preference is introduced by incorporating a subjective discount rate ρ into the utility function:

$$U(C, t) = \exp(-\rho t)\widetilde{U}(C, t) \tag{15}$$

The optimal conditions are given by:

$$\exp(-\rho t)\widetilde{U}_C(C^*, t) = J_W \tag{16}$$

$$(\alpha - r)W J_W + J_{WW} w^* W^2 \sigma^2 = 0 \tag{17}$$

where C^*, w^* are the optimal values.

Combining these conditions results in the so-called Hamilton-Jacobi-Bellman (HJB) equation, which is a partial differential equation for J,[10] one obtains:

$$U(G, t) + J_t + J_W (rW - G) - \frac{J_W^2}{J_{WW}} \frac{(\alpha - r)^2}{2\sigma^2} = 0 \tag{18}$$

subject to the boundary condition $J(W, T) = B(W, T)$. Merton (1971) solved the equation by restricting preferences, *assuming* that the utility function for the individual is a member of the Hyperbolic Absolute Risk Aversion (HARA) family of utility functions. The optimal C^* and w^* are then solved for as functions of J_W and J_{WW}, the riskless rate r, wealth W, and the parameters of the model (α and σ^2 of the price equation and the HARA parameters).

5.2 The Symmetries of the Model

We now derive the symmetries of the HJB equation (18) using the prolongation methodology. Two issues should be emphasized: (i) the symmetries are derived with *no* assumption on the functional form of the utility function except its concavity in C, a necessary condition for maximization; (ii) the optimal solution depends on the derivatives of the indirect utility function J, which, in turn, depends on wealth W and time t. The

[10]For a succint mathematical summary of HJB equations see Lions (1983).

idea is to derive transformations of t and W that would leave the optimality equation invariant. These transformations do not require imposing any restrictions on the end points, i.e., transversality conditions, of the type usually needed to obtain a unique solution to optimal control problems.[11]

In economic terms, this means that if wealth varies, because of taxation, the optimal solution remains invariant. The underlying interest in the invariance of the optimality equations is that we would like to have invariance of the structure of the solution across different levels of wealth.

5.2.1 Application of the Prolongation Methodology

In order to calculate the symmetries of the HJB equation (18), which is a p.d.e., we first calculate the infinitesimal generators of the symmetries, and then exponentiate these infinitesimal generators to get the symmetries themselves. An infinitesimal generator ν of the HJB equation has the following form, as in equation (8) above:

$$\nu = \xi(W,t,J)\frac{\partial}{\partial W} + \tau(W,t,J)\frac{\partial}{\partial t} + \phi(W,t,J)\frac{\partial}{\partial J} \tag{19}$$

Here ξ, τ, ϕ are functions of the variables W, t, J. The function J, as well as its partial derivatives, become variables in this method of derivation of the symmetries. In order to determine explicitly the functions ξ, τ, ϕ we prolongate the infinitesimal generator ν according to the prolongation formula of Olver (1993, page 110) and the equations thereby obtained provide the set of constraints satisfied by the functions ξ, τ, ϕ (see details in Olver (1993, pages 110-114), whose notation we use throughout).

The prolongation equation applied to ν yields:

$$\begin{aligned}&\left[r\xi J_W - \rho\tau e^{-\rho t}U(G(e^{\rho t}J_W)) + (rW - G(e^{\rho t}J_W))\phi^W + \phi^t\right] J_{WW}^2 \\ &+2A\phi^W J_W J_{WW} - A\phi^{WW} J_W^2 = 0\end{aligned} \tag{20}$$

where $\phi^W, \phi^t, \phi^{WW}$ are given by:

$$\phi^W = \phi_W + (\phi_J - \xi_W)J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t$$

$$\phi^t = \phi_t - \xi_t J_W + (\phi_J - \tau_t)J_t - \xi_J J_W J_t - \tau_J J_t^2$$

[11] The symmetries, however, do not restrict the optimal solution to be unique.

$$
\begin{aligned}
\phi^{WW} &= \phi_{WW} + (2\phi_{WJ} - \xi_{WW})J_W - \tau_{WW}J_t + (\phi_{JJ} - 2\xi_{WJ})J_W^2 \\
&\quad -2\tau_{WJ}J_W J_t - \xi_{JJ}J_W^3 - \tau_{JJ}J_W^2 J_t + (\phi_J - 2\xi_W)J_{WW} \\
&\quad -2\tau_W J_{Wt} - 3\xi_J J_W J_{WW} - \tau_J J_t J_{WW} - 2\tau_J J_W J_{Wt}
\end{aligned}
$$

Use (20) to derive the following restrictions:[12]

$$
\begin{aligned}
\phi_t &= 0 \\
\tau_t &= 0
\end{aligned} \tag{21}
$$

Thus we gather that $\tau = Constant$ and $\phi = Constant$ and we are left with the following equation for the ξ function:

$$
e^{\rho t}(r\xi - \xi_t - rW\xi_W)J_W + \xi_W G(e^{\rho t}J_W)e^{\rho t}J_W - \rho\tau U(G(e^{\rho t}J_W)) = 0 \tag{22}
$$

From this we deduce that $\xi_W = 0$ unless the following functional equation is satisfied

$$
G(e^{\rho t}J_W)e^{\rho t}J_W - \gamma U(G(e^{\rho t}J_W)) = 0 \tag{23}
$$

in which γ is a constant scalar. The last statement is of particular importance in the current context, as will be shown below.

We end up with the following constraints for the infinitesimal generators:

$$
\phi_t = 0 \tag{24}
$$

$$
\rho\tau = \phi \tag{25}
$$

$$
\xi_W = 0 \tag{26}
$$

The last equation holds true unless equation (23) is satisfied.

[12]The appendix shows the full derivation.

5.2.2 The Symmetries

The constraints, which we have derived, on the functions ξ, τ, ϕ and their derivatives completely determine the infinitesimal symmetries, which are given by:

Symmetry 1 $\phi_t = 0$

If $J(W,t)$ is a solution to the HJB equation, then so is $J(W,t) + k$ for any $k \in R$.

Symmetry 2 $\rho\tau = \phi$

If $J(W,t)$ is a solution of the HJB equation, so is $e^{-\rho\tau} J(W, t+\tau)$ for any $\tau \in R$.

Symmetry 3 $\xi_W = 0$

If $J(W,t)$ is a solution of the HJB equation, so is $J(W + ke^{rt}, t)$ for any $k \in R$.

For a general specification of the utility function, the HJB equation of the model admits only the above three symmetries. However, from the constraints above we also get that in case that the utility function satisfies the functional equation (23), and only in that case, there is an extra symmetry for the HJB equation.

Consider the implications of equation (23). For this we need first the following.

Lemma 1 *The functional equation*

$$G(x)x - \gamma U(G(x)) = 0 \tag{27}$$

where $G = (U')^{-1}$ *, is satisfied by a utility function* U *iff* U *is of the HARA form.*

Proof. Upon plugging in the equation a utility $U(x)$ of the HARA form we see the functional equation above is satisfied. Going the other way, after differentiating the equation with respect to x, we get the ordinary differential equation:

$$G'x + G - \varepsilon(xG') = 0$$

The solutions of this equation form the HARA class of utility functions. Then we take the inverse function to get U' and after integration we get that U is of the HARA form.
■

When the functional equation (23) holds true and Lemma 1 is relevant, the exponentiation of the infinitesimal generators yields a fourth symmetry as follows.

Symmetry 4
If $J(W,t)$ is a solution, then so is $e^{k\gamma}J(e^{-k}\{W+\frac{(1-\gamma)\eta}{\beta r}\}-\eta\frac{(1-\gamma)}{\beta r},t)$ for any $k\in R$.

In words, this fourth symmetry says that if $J(W,t)$ is a solution then a linear function of J is also a solution. Calculation of the solutions to the functional equation in Lemma 1 shows that only utility functions of the HARA class satisfy it. The HARA function is expressed as follows:

$$\widetilde{U}(C)=\frac{1-\gamma}{\gamma}\left(\frac{\beta C}{1-\gamma}+\eta\right)^{\gamma} \tag{28}$$

The special case of the CRRA function $\frac{C^{\gamma}}{\gamma}$ has $\beta=(1-\gamma)^{-\frac{(1-\gamma)}{\gamma}},\eta=0$. The cases of logarithmic utility ($\ln C$) and exponential utility ($-e^{-\beta C}$) are limit cases.

We note a distinction between two concepts: (i) restrictions on the utility function for scale invariance of the preference relation, which is a topic that is not treated here, and was dealt with in seminal work by Skiadas (2009, Chapters 3 and 6, 2013); (ii) scale invariance of an optimality equation, in the form of a differential equation, which is the object of inquiry here.

5.3 Economic Interpretation

The first three symmetries formulate "classical" principles of utility theory and there are no new insights gained from them.[13]As noted, for a general specification of the utility function, the HJB equation of the model admits only the above three symmetries. But if equation (23) is satisfied, the fourth symmetry above places restrictions on the utility function, and is the main point of interest here. Because k is completely arbitrary any multiplicative transformations of W, i.e., $e^{-k}W$, apply.[14] Such transformations are the most natural ones to consider when thinking of taxation policy. Thus, this symmetry states the following: the optimum, expressed by the J function, i.e., maximum expected life-time utility, will remain invariant under multiplicative transformations of wealth W *if and only if* HARA utility is used. Note well that HARA utility is implied by this

[13]Symmetry 1 represents a formulation of the idea that utility is ordinal and not cardinal; symmetry 2 expresses the property that displacement in calendar time does not change the optimal solution; symmetry 3 expresses a property with respect to W that is similar to the property of aymmetry 2 with respect to t : if the solution is optimal for W then it is also optimal for an additive re-scaling of W; the term e^{rt} keeps the additive k constant in present value terms.

[14]If, together with the multiplicative transformation, there is also an additive transformation of W, expressed by the term $e^{-k}\left(\frac{(1-\gamma)\eta}{\beta r}\right)-\eta\frac{(1-\gamma)}{\beta r}$, then it is further restricted by the parameters of the HARA utility function, and features the same arbitrary constant k used for the multiplicative transformation.

symmetry, not assumed a-priori. This does not imply, though, that there is a unique such J but it does express a property of any J which solves the HJB equation.

It should be emphasized that Symmetry 4 establishes the conditions for invariance without solving the model. The analysis of Lie symmetries does not necessitate the assumption of particular forms for the utility functions, solving the optimality equations, and comparing the optimal solutions across the assumed functions. Hence, Symmetry 4 establishes the HARA requirement without assuming any functional form for the utility function and without solving the model in closed form.

5.4 The Implications for Structural Economic Policy Analysis

Symmetry 4 has the following implications for structural models used for economic policy analysis.

(i) *The form of the utility function and invariance of the optimal solution.* The HARA form is determined by the symmetry. The idea, then, is that there is an interdependence between the functional form of preferences (the form of the utility function) and the requirement that the optimal solution will remain invariant under multiplicative wealth transformations. This kind of invariance underpins empirical undertakings as discussed in Section 3 above.

(ii) *Scale Invariance and Linear Optimal Rules.* In Merton (1971, p. 391) the following theorem is presented and proved:

> THEOREM III. Given the model specified...$C^* = aW + b$ and $w^*W = gW + h$ where a, b, g, and h are, at most, functions of time if and only if $U(C,t) \subset HARA(C)$.

The result we obtain above – Symmetry 4 – can be stated as follows:

Theorem 2 *Given the model specified in this section, then Symmetry 4 (the scaling symmetry) is satisfied if and only if* $U(C,t) \subset HARA(C)$.

Combining the last theorem with Merton's theorem III, we get:

Corollary 3 *Given the model specified in this section, then* $C^* = aW + b$ *and* $w^*W = gW + h$ *where* a, b, g, *and* h *are, at most, functions of time, if and only if Symmetry 4 (the scaling symmetry) is satisfied.*

The proof of this corollary does not necessitate a specific solution to the HJB equation in the model, and in any case it is impossible to give a precise solution when the utility function is not specified. This means that wealth scale invariance implies linear optimal solutions to the control variables (C^*, w^*) and linear optimal rules imply scale invariance. Scale invariance determines the relevant linear parameters of optimal behavior.

This is not simply a re-statement of Merton's (1969, 1971) results. The latter papers have assumed HARA utility and then solved the HJB equation.[15] Here Symmetry 4 shows that utility *has* to be HARA, so that the consumer-investor problem be invariant for economic plausibility and for structural empirical investigation. This is established even without solving the HJB equation.

Such structural, invariant, linear relations are very useful for policy design and evaluation.

(iii) *Comparisons Across Consumers/Investors.* If we know that we can compare the outcome of two different consumers/investors as a linear function of the ratio of their wealth stocks, then necessarily the utility function of the agents is of the HARA form. This allows for interpersonal comparisons for policy purposes, for example, when taxation is a function of the level of wealth W.

(iv) *Aggregation and Equilibrium Modelling.* The linearity facilitates aggregation and the use of representative agent modelling. It is highly important for the construction of an equilibrium model, such as the seminal Merton (1973) intertemporal CAPM model, which embeds this set-up. Once more, this greatly facilitates tax policy analysis.

(v) *Structural Econometrics*. Consider structural econometrics in the context of this model. We have the consumer-investor optimal behavior functions, which are the optimal solutions to the HJB equations in the Merton (1969,1971) model, If policy changes wealth through taxation, then equations (16)-(17) offer a structural model, conforming the afore cited definitions by Heckman and Vytlacil (2007). This model can be estimated across agents and over time for γ, β, η, and the parameters of the stochastic process.

6 Possible Applications at the Research Frontier

The preceding analysis has demonstrated the use of Lie symmetries as a tool to derive invariance restrictions in economic optimization problems, thereby facilitating the use of structural models. We have used a model which has a closed form and well-known

[15] See, for example, pp. 388-391 in Merton (1971).

solution. In what follows, we point to more complicated models, some of which have no closed-form solutions or well-defined functional forms. These models are, however, amenable to the same analysis. Note that there are likely to be many different problems that would yield restrictions of the type explored here. We consider the following models, which are at the research frontier, noting policy issues for each.

Counter-factual equivalence. The use of Lie symmetries can greatly extend the scope of models examined using the principle of counterfactual equivalence, suggested by Beraja (2020) for macroeconomic models. His idea is as follows: counterfactuals in structural models are a leading way to analyze policy rule changes, because they are immune to the Lucas Critique. But there are issues as to the appropriate choice of model primitives for these structural models. For example: how do the effects of policy change under variations in the policy rule for different primitives? How does the modeler decide on these primitives? Beraja (2020) proposes methods to deal with these issues. The methods rest on the insight that many models, which are well approximated by a linear representation, are both observationally equivalent under a benchmark policy and yield an identical counterfactual equilibrium under alternative policy. These are called "counter-factually equivalent models." They can be found through analysis of linear restrictions. One can then know which models will be observationally equivalent under both benchmark and alternative (counterfactual) policy rules, and which will not be. As an example, consider one application examined by Beraja (2020). He shows that search models are counter-factually equivalent across those DMP-type models, which change the primitives of firms' incentives or the job creation technology structure. But models, which change the primitives of wage setting and bargaining, are not counter-factually equivalent.

The algebraic method of Lie symmetries can be used in this context as follows. Write the relevant model equilibrium equations as differential equations. Note that these do not have to be linear, as in Beraja's case. Derive the Lie symmetries for these differential equations. Use the symmetries to identify restrictions on the relevant structural model primitives such that the model remains invariant under policy rules variations. Those models that satisfy these restrictions are counter-factually equivalent.

Heterogenous Agents and Policy Regimes. The problem of modelling optimal behavior in an economy with heterogenous agents is amenable to such analysis. This kind of problem is an important one in complex DSGE models with heterogeneous agents, which have become pervasive in business cycle modelling; see Krueger, Mitman, and Perri (2016) for a recent overview. Invariance issues are highly pertinent in these models. Thus, for example, Chang, Kim, and Schorfheide (2013) simulate data from a heterogeneous-agents economy, under various policy regimes, and then estimate an approximating representative-agent model. They find that preference and technology parameter estimates of the representative-agent model are not invariant to policy changes. Indeed they find that the bias in the representative-agent model's policy predictions is large. They conclude that "since it is not always feasible to account for heterogeneity explicitly, it is important to recognize the possibility that the parameters of a highly aggregated model may not be invariant with respect to policy changes." Lie symmetries can provide conditions for aggregator functions and restrictions on the multitude of functions in the model, such as the utility, production, or costs functions. The latter can include price, labor, capital, and financial frictions. Beyond providing restrictions, the symmetries inform the researcher on the properties of the solution, including cases whereby closed-form solutions do not exist. These conditions and restrictions may be very useful in generating insights on key issues, such as the marginal propensity to consume across heterogenous consumers, the response of consumption behavior to monetary policy and to fiscal policy, and the response of heterogenous firms to these policies. As mentioned above for the Merton (1973) intertemporal CAPM model, equilibrium characterizations are facilitated by the invariant structure uncovered by the symmetries. Kaplan, Moll, and Violante (2018) show that using continuous time is natural in the context of the Heterogenous Agents New Keynesian (HANK) model. Hence the use of differential equations for optimality relations, including the HJB equation akin to the one examined above is possible,[16]and amenable to the Lie symmetries analysis.

Income and Wealth Distributions. A related class of heterogenous agents models studies income and wealth distributions. Achdou, Han, Lasry, Lions, and Moll (2020) make an important contribution. These authors boil the model down to systems of two coupled partial differential equations, the Hamilton-Jacobi-Bellman (HJB) equation for the optimal choices of a single atomistic individual, who takes the evolution of the distribution, and hence prices, as given; and the Kolmogorov Forward (KF) equation characterizing the evolution of the distribution, given optimal choices of individuals. In complementarity with the mathematical tools proposed by Achdou et al (2020), Lie symmetries can be used to provide the entire set of relevant restrictions on the HJB and KF equations.

Time-invariance of Preferences. This issue has been studied a number of times over the past two decades, starting from Barro (1999) and going all the way to recent treatments, such as Millner and Heal (2018), who provide an overview. Lie symmetries provide restrictions on the time preference function. Halevy (2015) points out that

[16]See the discussion in Kaplan, Moll, and Violante (2018, pp. 702-3, 709-710, and Appendix B.1).

one needs to distinguish between stationarity, time invariance, and time consistency of preferences. He shows that any two of these properties imply the third. Hence, time invariance is an important feature related to the time consistency of preferences. There has been great interest in Macroeconomics in the latter issue, predominantly for consumption decisions and for policymaker plans, monetary and fiscal, including debt. Lie symmetries can provide invariance restrictions both on the time dimension and on the cross-sectional dimension, as done above, in Symmetries 2,3, and 4. Rather than assume certain functional forms of time preferences, these could be derived using the tools presented above. The idea, here too, is to employ invariance restrictions based on economic reasoning.

Indeterminacy, bubbles, and sunspots. Lie symmetries can address issues of indeterminacy, bubbles, and sunspots, such as those that arise in macroeconomic models. These topics have re-emerged given recent empirical experience with various "bubbles" phenomena and new modelling; see Miao (2014, 2016). For a GE model with flexible prices, see Pintus (2006, 2007) and for an OLG model with nominal rigidities, see Galí (2014). By deriving the Lie symmetries of the optimality equations of the model (such as equations (8) in Pintus (2006) or in Pintus (2007), or equations (3)-(6) in Galí (2014) formulated in continuous time), one obtains conditions relating the production and utility functions to agents' optimal behavior, as done in this paper for the utility function and the HJB equation of the Merton (1969, 1971) model. Thereby the analysis would yield restrictions that need to be placed on these functions. The restrictions, required to insure invariance of the optimality equations, would shed light on the problems of indeterminacy.

7 Conclusions

This paper has shown how the algebraic methodology of Lie symmetries can be useful for formulating structural models for the design and evaluation of policy. At the heart of the analysis is the concept of invariance. This is so as one wants to formulate a model featuring parameters, which are policy invariant. Lie symmetries is a methodology long used in other sciences, such as Physics. It seems only natural to apply it in Economics, this paper showing how to do it, and suggesting a number of important applications out of a large set of possible models.

References

[1] Achdou, Yves, Francisco J. Buera, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, 2014." Partial Differential Equation Models in Macroeconomics," **Philosophical Transactions of the Royal Society** A 372: 20130397.

[2] Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll, 2020. "Income and Wealth Distributions in Macroeconomics: A Continuous-Time Approach," **Review of Economic Studies**, forthcoming.

[3] Athey, Susan and Guido W. Imbens, 2017. "The State of Applied Econometrics: Causality and Policy Evaluation," **Journal of Economic Perspectives** 31,2, 3–32.

[4] Barro, Robert J., 1999. "Ramsey Meets Laibson in the Neoclassical Growth Model," **The Quarterly Journal of Economics** 114, 4, 1125-1152.

[5] Beraja, Martin, 2020. "A Semi-structural Methodology for Policy Counterfactuals" working paper. downloadable at https://economics.mit.edu/files/17173

[6] Chang, Yongsung, Sun-Bin Kim and Frank Schorfheide, 2013. "Labor Market Heterogeneity, Aggregation, and Policy (In)variance of DSGE Model Parameters" **Journal of the European Economic Association** 11,s1,193-220.

[7] Chen, An, Antoon Pelsser, and Michel Vellekoop, 2011. "Modeling Non-Monotone Risk Aversion Using SAHARA Utility Functions," **Journal of Economic Theory** 146, 2075–2092.

[8] Duffie, Darrell, 2003. "Intertemporal Asset Pricing Theory" in G.M.Constantinides, M. Harris and R. Stulz (eds.) **Handbook of the Economics of Finance** Vol. 1B, 639–742.

[9] Filho, Tarcísio M. Rocha and Annibal Figueiredo, 2011. "[SADE] a Maple Package for the Symmetry Analysis of Differential Equations," **Computer Physics Communications**, 182, 467–476.

[10] Frisch, Ragnar, 1938. "Statistical vs Thoretical Relations in Economic Macrodynamics," submitted to the Business Cycle Conference in Cambridge UK; republished in D.F. Hendry and M.S. Morgan (eds., 1995) **The Foundations of Economic Analysis,** Cambridge, UK.

[11] Galí, Jordi, 2014. “Monetary Policy and Rational Asset Price Bubbles,” **American Economic Review** 104(3): 721–752.

[12] Haavelmo, Trygve, 1943. “The Statistical Implications of a System of Simultaneous Equations.” **Econometrica** 11(1), 1–12.

[13] Halevy, Yoram, 2015. “Time Consistency: Stationarity and Time Invariance, ” **Econometrica** 83, 1, 335–352.

[14] Hansen, Lars, 2014. “Nobel Lecture: Uncertainty Outside and Inside Economic Models,” **Journal of Political Economy** 122,5, 945-987.

[15] Hansen, Lars P. and Thomas J. Sargent, 1980. “Formulating and Estimating Dynamic Linear Rational Expectations Models.” **Journal of Economic Dynamics and Control** 2(1), 7–46.

[16] Heckman, James J., 2008.“ Econometric Causality” **International Statistical Review** 76(1), 1–27.

[17] Heckman, James J. and Rodrigo Pinto, 2014. “Causal Analysis After Haavelmo,” **Econometric Theory**, 1-37.

[18] Heckman, James J. and Edward J. Vytlacil, 2007. “Econometric Evaluation of Social Programs, part I: Causal Models, Structural Models and Econometric Policy Evaluation.” In J. Heckman & E. Leamer (eds.), **Handbook of Econometrics, vol. 6B,** 4779–4874, Elsevier, Amsterdam.

[19] Kaibe, Bosiu C. and John G. O’Hara, 2019. “Symmetry Analysis of an Interest Rate Derivatives PDE Model in Financial Mathematics,” **Symmetry** 11, 1056.

[20] Kannai,Yakar, 2004. “When is Individual Demand Concavifiable?” **Journal of Mathematical Economics** 40, 59–69.

[21] Kannai,Yakar, 2005. “Remarks Concerning Concave Utility Functions on Finite Sets,” **Economic Theory** 26, 333–344.

[22] Kaplan, Greg, Benjamin Moll, and Giovanni L. Violante, 2018. “Monetary Policy According to HANK,” **American Economic Review** 108,3, 697–743.

[23] Krueger, Dirk, Kurt Mitman, and Fabrizio Perri, 2016. “Macroeconomics and Household Heterogeneity,” Chapter 11 in J. Taylor and H.Uhlig (eds.) **Handbook of Macroeconomics Vol. 2**, 843-921, North-Holland, Amsterdam.

[24] Lions, Pierre-Louis, 1983."On the Hamilton-Jacobi-Bellman Equations," **Acta Applicandae Mathematica** 1, 1, pp 17–41.

[25] Low, Hamish and Costas Meghir, 2017. "The Use of Structural Models in Econometrics," **Journal of Economic Perspectives** 31, 2, 33–58

[26] Lucas, Robert E., 1976. "Econometric Policy Evaluation: A Critique," **Carnegie-Rochester Conference on Public Policy**, 1, 19-46.

[27] Merton, Robert C., 1969. "Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case," **Review of Economics and Statistics** 51, 247-57.

[28] Merton, Robert C., 1971. "Optimum Consumption and Portfolio Rules in a Continuous Time Model," **Journal of Economic Theory** 3, 373-413.

[29] Merton, Robert C., 1973. "An Intertemporal Capital Asset Pricing Model," **Econometrica** 41, 867-87.

[30] Merton, Robert C., 1990. **Continuous Time Finance**, Blackwell, Cambridge, MA.

[31] Miao, Jianjun, 2014. "Introduction to Economic Theory of Bubbles," **Journal of Mathematical Economics** 53, 130–136.

[32] Miao, Jianjun, 2016. "Introduction to Economic Theory of Bubbles II," **Journal of Mathematical Economics** 65, 139–140.

[33] Millner, Antony and Geoffrey Heal, 2018. "Time Consistency and Time Invariance in Collective Intertemporal Choice" **Journal of Economic Theory** 176, 158-169.

[34] Olver, Peter J., 1993. **Applications of Lie Groups to Differential Equations**, 2nd edition, Springer-Verlag, New York.

[35] Olver, Peter J., 1999. **Classical Invariant Theory**, Cambridge University Press,Cambridge.

[36] Olver, Peter J., 2012. **Lie Groups and Differential Equations**, Lecture Notes, available at http://www-users.math.umn.edu/~olver/sm.html

[37] Pintus, Patrick A., 2006. "Indeterminacy with Almost Constant Returns to Scale: Capital– Labor Substitution Matters," **Economic Theory** 28, 633–649.

[38] Pintus, Patrick A., 2007. "Local Determinacy with Non-Separable Utility," **Journal of Economic Dynamics & Control** 31, 669–682

[39] Sato, Ryuzo, 1981. **Theory of Technical Change and Economic Invariance: Application of Lie Groups**, Academic Press, New York. Reprinted with amendments, 1999, Edward Elgar Publishing Ltd., Cheltenham.

[40] Sato, Ryuzo, and Rama V. Ramachandran (eds.), 1990. **Conservation Laws and Symmetries**, Kluwer Academic Publishers, Boston.

[41] Sato, Ryuzo, and Rama V. Ramachandran, 2014. **Symmetry and Economic Invariance,** Second Enhanced Edition, Springer, Japan.

[42] Sinkala, W., P.G.L. Leach, and J.G. O'Hara, 2008a. "Invariance Properties of a General Bond-Pricing Equation," **Journal of Differential Equations** 244, 2820-2835.

[43] Sinkala, W., P.G.L. Leach, and J.G. O'Hara, 2008b. "Zero-Coupon Bond Prices in the Vasicek and CIR Models: Their Computation as Group-Invariant Solutions," **Mathematical Methods in the Applied Sciences** 31, 665–678.

[44] Skiadas, Costis, 2009. **Asset Pricing Theory**, Princeton University Press, Princeton.

[45] Skiadas, Costis, 2013. "Scale-Invariant Uncertainty-Averse Preferences and Source-Dependent Constant Relative Risk Aversion," **Theoretical Economics** 8, 59-93.

[46] Vu, Khai.T., G.F. Jefferson, and John Carminati, 2012. "Finding Higher Symmetries of Differential Equations Using the MAPLE Package DESOLVII" **Computer Physics Communications** 183,4, 1044–1054.

[47] Weil, Philippe, 1989. "The Equity Premium Puzzle and the Risk Free Rate Puzzle," **Journal of Monetary Economics** 24, 401-421.

[48] Weil, Philippe, 1990. "Nonexpected Utility in Macroeconomics," **The Quarterly Journal of Economics,** 105,1, 29-42.

Appendix
Full Derivation of the Symmetries

The prolongation equation applied to ν yields:

$$\begin{aligned} & \left[r\xi J_W - \rho\tau e^{-\rho t} U(G(e^{\rho t} J_W)) + (rW - G(e^{\rho t} J_W))\phi^W + \phi^t \right] J_{WW}^2 \\ & +2A\phi^W J_W J_{WW} - A\phi^{WW} J_W^2 \\ = & \ 0 \end{aligned} \tag{29}$$

where $\phi^W, \phi^t, \phi^{WW}$ are given by:

$$\phi^W = \phi_W + (\phi_J - \xi_W) J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t$$

$$\phi^t = \phi_t - \xi_t J_W + (\phi_J - \tau_t) J_t - \xi_J J_W J_t - \tau_J J_t^2$$

$$\begin{aligned} \phi^{WW} = & \ \phi_{WW} + (2\phi_{WJ} - \xi_{WW}) J_W - \tau_{WW} J_t + (\phi_{JJ} - 2\xi_{WJ}) J_W^2 \\ & -2\tau_{WJ} J_W J_t - \xi_{JJ} J_W^3 - \tau_{JJ} J_W^2 J_t + (\phi_J - 2\xi_W) J_{WW} \\ & -2\tau_W J_{Wt} - 3\xi_J J_W J_{WW} - \tau_J J_t J_{WW} - 2\tau_J J_W J_{Wt} \end{aligned}$$

Plugging these expressions in the prolongation formula applied to the HJB equation (18) yields:

$$
\begin{aligned}
& r\xi J_W - \rho\tau e^{-\rho t} U(G(e^{\rho t} J_W)) \qquad (30)\\
& +(rW - G(e^{\rho t} J_W))(\phi_W + (\phi_J - \xi_W) J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t)\\
& +(\phi_t - \xi_t J_W + (\phi_J - \tau_t) J_t - \xi_J J_W J_t - \tau_J J_t^2) J_{WW}^2\\
& +2A(\phi_W + (\phi_J - \xi_W) J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t) J_W J_{WW}\\
& -A \begin{pmatrix} \phi_{WW} \\ +(2\phi_{WJ} - \xi_{WW}) J_W \\ -\tau_{WW} J_t \\ +(\phi_{JJ} - 2\xi_{WJ}) J_W^2 \\ -2\tau_{WJ} J_W J_t \\ -\xi_{JJ} J_W^3 \\ -\tau_{JJ} J_W^2 J_t \\ +(\phi_J - 2\xi_W) J_{WW} \\ -2\tau_W J_{Wt} \\ -3\xi_J J_W J_{WW} \\ -\tau_J J_t J_{WW} \\ -2\tau_J J_W J_{Wt} \end{pmatrix} J_W^2\\
= {} & 0
\end{aligned}
$$

Note that the variables J_W, J_{WW}, J_{Wt}, J_t are algebraically independent. This implies that the coefficients of the different monomials in those variables are equal to zero. We therefore proceed as follows.

(i) We first look at the different monomials in the above equation in which J_{WW} does not appear. Equating the coefficients of these monomials to 0 implies that:

$$
\begin{aligned}
\tau_J &= \tau_W = 0 \qquad (31)\\
\xi_{JJ} &= 0\\
\phi_{WW} &= 0\\
\phi_{JJ} - 2\xi_{JW} &= 0\\
2\phi_{WJ} - \xi_{WW} &= 0
\end{aligned}
$$

(ii) Next we look at monomials in which J_{WW} appears in degree one. This gives (noting that $r \neq \alpha$ implies that $A \neq 0$) the following equation (in which we gathered only those monomials with their coefficients):

$2(\phi_W + (\phi_J - \xi_W)J_W - \xi_J J_W^2)J_W J_{WW} + 3\xi_J J_W^3 J_{WW} - (\phi_J - 2\xi_W)J_{WW}J_W^2 = 0$

From this we deduce that

$$\begin{aligned} \phi_W &= 0 \\ \xi_J &= 0 \\ \phi_J &= 0 \end{aligned} \tag{32}$$

(iii) Now we look at the monomials containing J_{WW}^2 which give the following equation:

$$r\xi J_W - \rho\tau e^{-\rho t}U(G(e^{\rho t}J_W)) - (rW - G(e^{\rho t}J_W))\xi_W J_W + \phi_t - \xi_t J_W - \tau_t J_t = 0 \tag{33}$$

From which we deduce that

$$\begin{aligned} \phi_t &= 0 \\ \tau_t &= 0 \end{aligned} \tag{34}$$

From all the constraints above on the functions ξ, τ, ϕ we gather so far that $\tau = Constant$ and $\phi = Constant$ and we are left with the following equation for the ξ function:

$$e^{\rho t}(r\xi - \xi_t - rW\xi_W)J_W + \xi_W G(e^{\rho t}J_W)e^{\rho t}J_W - \rho\tau U(G(e^{\rho t}J_W)) = 0 \tag{35}$$

From this we deduce that $\xi_W = 0$ unless the following functional equation is satisfied

$$G(e^{\rho t}J_W)e^{\rho t}J_W - \gamma U(G(e^{\rho t}J_W)) = 0 \tag{36}$$

in which γ is a constant scalar. The last statement is of great importance in the current context, as will be shown below.

We end up with the following constraints for the infinitesimal generators:

$$\phi_t = 0 \tag{37}$$

$$\rho\tau = \phi \tag{38}$$

$$\xi_W = 0 \tag{39}$$

The last equation holds true unless equation (23) is satisfied.